Large-Deflection and Nonlinear Vibration of Multilayered Sandwich Plates

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A quadratic rectangular finite element including all nonlinear terms in the strain-displacement relations is used to study the large-deflection and nonlinear vibration behavior of multilayered sandwich plates. Results are presented for simply supported and clamped boundary conditions and for a combination of these conditions. In general, it is observed that a plate with clamped boundary yields the least nonlinearity and that the nonlinear effect is predominant in the case of plates with a greater number of layers. Keeping the total thickness of the faces and that of cores constant in each case, a seven-layered sandwich is found to be more flexible compared to a five-layered one.

Introduction

THE main purpose of a sandwich construction is to achieve a stiff, lightweight structure, which is accomplished by using thin, high-strength, high-modulus material for the facings and low-density material for the core. The function of the core is to separate the facings so that they are far enough from the middle surface to carry an appreciable bending load. In an unsymmetrically laminated multicore sandwich, a bending-stretching coupling occurs that is not present in a symmetrically laminated sandwich structure.

Research on multilayered sandwich plates is usually limited to linear analysis in the existing literature.[1] A bending theory for multicore plates was presented by Liaw and Little,[2] which was extended to buckling by Wong and Salama.[3] Azar[4] derived the governing equations for a multicore sandwich with thin anisotropic facings. The free vibrations of multicore orthotropic thin sandwich plates for the linear case was the subject of research of Ref. 5.

An excellent review of nonlinear vibration analysis of plates and shell problems is given by Leissa.[6] The effects of shear and rotary inertia on the large-amplitude vibrations were studied by Celep[7] for rectangular plates and by Sathyamoorthy and Chia[8] for composite plates of parallelogram planform. Yamaki and Chiba[9] investigated the effects of geometric imperfections on the linear and nonlinear vibration behavior of isotropic homogeneous rectangular plates. Recently, Hui[10] pointed out that the presence of small-imperfection amplitudes would alter the character of composite plates from inherently hard-spring to soft-spring behavior.

The nonlinear bending and vibration behavior of multilayered sandwich plates has not been reported previously and forms the subject matter of the present investigation. Results are presented for five- and seven-layered configurations under different boundary conditions.

Formulation

Consider a multilayered sandwich plate, as shown in Fig. 1, having n stiff faces and $n-1$ soft cores. The displacement field of such a plate can be assumed as

$$u(x,y,z,t) = u^0(x,y,t) + z\psi_x(x,y,t)$$
$$v(x,y,z,t) = v^0(x,y,t) + z\psi_y(x,y,t)$$
$$w(x,y,z,t) = w^0(x,y,t) \tag{1}$$

where t is the time, u, v, and w are the displacements in the x, y, and z directions, respectively; u^0, v^0, and w^0 are the associated midplane displacements; and ψ_x and ψ_y are the slopes in the x-z and y-z planes due to bending. Assuming that the plate is moderately thick, the nonlinear strain-displacement relations can be written as

$$\epsilon_x = u,_x + z\psi_{x,x} + \tfrac{1}{2}[u,_x^2 + v,_x^2 + w,_x^2]$$
$$\epsilon y = v,_y + z\psi_{y,y} + \tfrac{1}{2}[u,_y^2 + v,_y^2 + w,_y^2]$$
$$\epsilon xy = u,_y + v,_x + z(\psi_{x,y} + \psi_{y,x}) + [u,_x u,_y + v,_x v,_y + w,_x w,_y]$$
$$\epsilon_{zz} = \psi_x^2 + \psi_y^2, \qquad \epsilon_{xz} = \psi_x + w,_x + [u,_x\psi_x + v,_x\psi_y]$$
$$\epsilon_{yz} = \psi_y + w,_y + [u,_y\psi_x + v,_y\psi_y] \tag{2}$$

Since the relations are based on plane stress assumptions, ϵ_{zz} can be neglected.

In the absence of the body forces, the total potential energy equation, after neglecting the body moments and surface shearing forces, takes the form

$$\pi = \frac{1}{2}\int_{t_1}^{t_2}\int_0^a\int_0^b \{[R(u,_t^2 + v,_t^2 + w,_t^2) + 2S(u,_t\psi_{x,t} + v,_t\psi_{y,t})$$
$$+ I(\psi_{x,t}^2 + \psi_{y,t}^2)] - [N_1\epsilon_1 + N_2\epsilon_2 + N_6\epsilon_6 + Q_1\epsilon_4$$
$$+ Q_2\epsilon_5 + M_1K_1 + M_2K_2 + M_6K_6]\}\mathrm{d}x\mathrm{d}y\mathrm{d}t \tag{3}$$

where R, S, and I are normal, coupled normal rotary, and rotary inertia coefficients and are given by

$$(R,S,I) = \sum_m \int_{z_m}^{z_{m+1}} \zeta^{(m)}(1,z,z^2)\mathrm{d}z$$

$\zeta^{(m)}$ is the material density of the mth layer of the sandwich, q is the transversely distributed force, and N_i, Q_i, and M_i are

Received June 17, 1985; revision received March 18, 1986.

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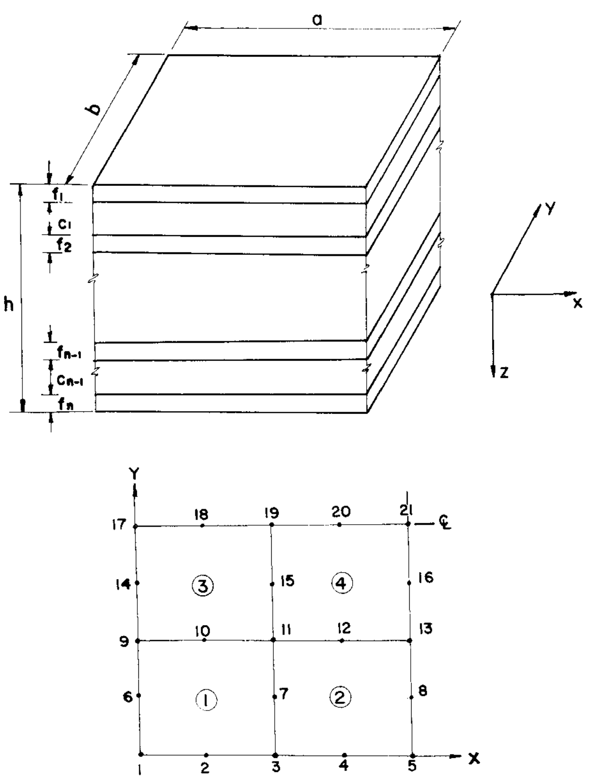

Fig. 1 Geometry of a multilayered sandwich plate and finite element idealization.

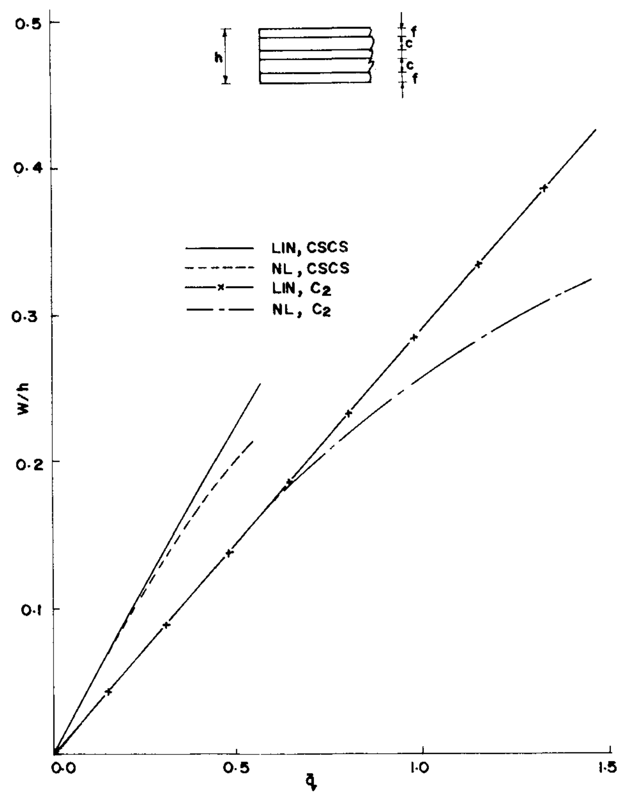

Fig. 2 Central deflection vs load (five-layered case): $f/h=0.02325$; $c/h=0.465$; $E_f=7031$ kg/mm^2; core: $G_{xz}=G_{yz}=21.0931$ kg/mm^2, $a/b=1.0$.

the stress and moment resultants and are given by

$$\begin{Bmatrix} N_i \\ \hline M_i \end{Bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix} \begin{Bmatrix} \epsilon_j \\ \hline K_j \end{Bmatrix} \qquad j,\, i=1,2,6$$

and

$$\begin{Bmatrix} Q_1 \\ \hline Q_2 \end{Bmatrix} = \begin{bmatrix} K_4^2 A_{44} & K_4 K_5 A_{45} \\ K_4 K_5 A_{54} & K_5^2 A_{55} \end{bmatrix} \begin{Bmatrix} \epsilon_4 \\ \epsilon_5 \end{Bmatrix}$$

where

$$(A_{ij},\, B_{ij},\, D_{ij}) = \sum_m \int_{z_m}^{z_{m+1}} Q_{ij}^{(m)}\,(1,z,z^2)\mathrm{d}z$$

and z_m denotes the distance from the midplane to the lower surface of the mth layer. K_4 and K_5 are the shear correction factors given as 0.640156 and 0.8315607, respectively, for a sandwich by Whitney.[11]

Let the plate region be divided into a finite number of quadratic elements such as those shown in Fig. 1. The generalized displacements (u, v, w, ψ_x, and ψ_y) in each element are taken as

$$u = \sum_{i=1}^{\gamma} u_i \phi_i^1, \quad v = \sum_{i=1}^{\gamma} v_i \phi_i^1, \quad w = \sum_{i=1}^{s} w_i \phi_i^2,$$

$$\psi_x = \sum_{i=1}^{p} \psi_{xi} \phi_i^3, \quad \psi_y = \sum_{i=1}^{p} \psi_{yi} \phi_i^3 \qquad (4)$$

where ϕ_i^α ($\alpha=1,2,3$) is the interpolation function corresponding to the ith node in the element and γ, s, and p are the number of degrees of freedom for each variable. For simplicity, we take $\phi_i^1=\phi_i^2=\phi_i^3$ and $\gamma=s=p$.

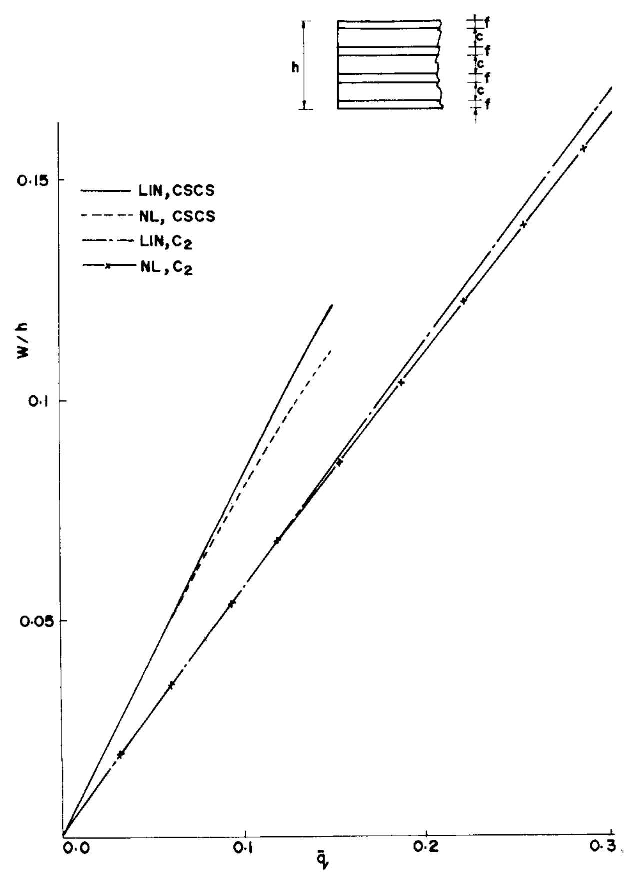

Fig. 3 Central deflection vs load (seven-layered case): $f/h=0.015625$; $c/h=0.3125$; $E_f=7031$ kg/mm^2; core: $G_{xz}=G_{yz}=21.0931$ kg/mm^2, $a/b=1.0$.

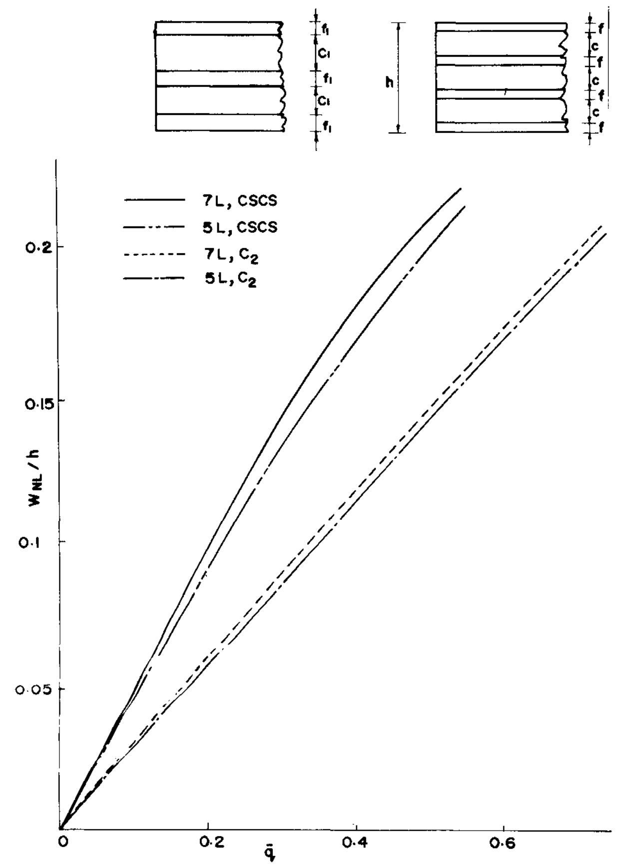

Fig. 4 Comparison of five- and seven-layered cases: $f1/h=0.02325$, $c1/h=0.465$, $f/h=0.01744$; $c/h=0.31$; $E_{f1}=E_f=7031$ kg/mm^2, $a/b=1.0$; core: $G_{xz}=G_{yz}=21.093$ kg/mm^2.

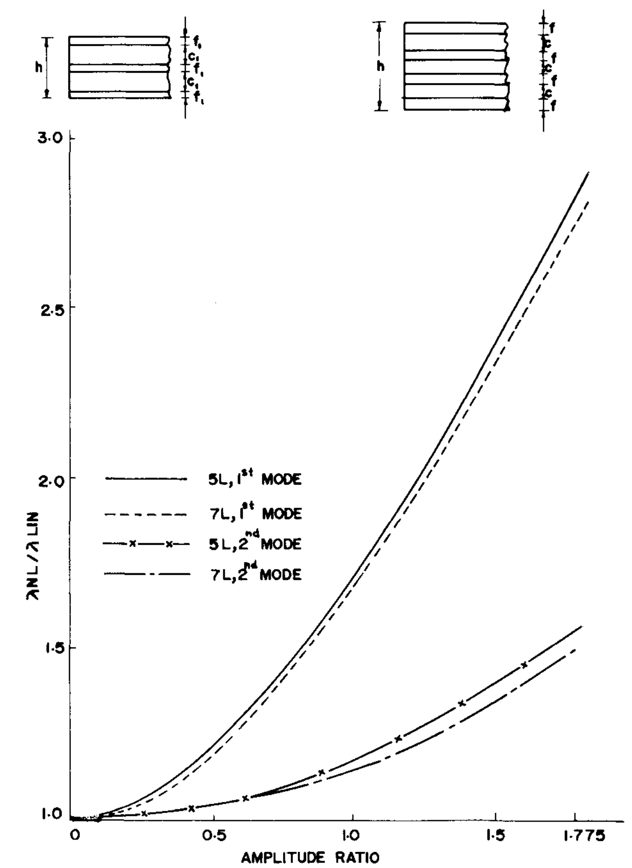

Fig. 5 Variation of frequency ratio with amplitude ratio: $f1/h=0.03887$; $c1/h=0.4417$; $f/h=0.2473$; $c/h=0.3$; $E_{f1}=E_f=7031$ kg/mm^2, $a/b=1.0$; core: $G_{xz}=13.71$ kg/mm^2, $G_{yz}=5.27$ kg/mm^2, $\zeta_{f1}=\zeta_f=2.8225\times10^{-10}$ kg s^2/mm^4, $\zeta_c=1.2423\times10^{-11}$ kg s^2/mm^4.

Table 1 Comparison of central deflection

		Linear deflection, mm	
Sl. no.	Case	Present value	Ref. 13
1	5-layered (SS)	0.017272	0.01778
2	5-layered (C2)	0.0023876	—
3	7-layered (SS)	0.009144	—
4	7-layered (C2)	0.0011176	—

Assuming serendipity polynomials as given in Ref. 12, using Eq. (4), and substituting in the variational form of Eq. (3), one gets

$$[K]_e\{\Delta\}_e=\omega^2[M]_e\{\Delta\}_e+\{F\}_e \tag{5}$$

where $[K]_e$ is the stiffness matrix, $[M]_e$ the mass matrix, and $\{F\}_e$ the load, and $\{\Delta\}_e$ the displacement vectors for each element.

Numerical Results

Results are presented in this section for five- and seven-layered sandwich plates. For the st:tic bending case, Eq. (5) takes the form

$$[K]_e\{\Delta\}_e=\{F\}_e \tag{6}$$

A Gaussian quadrature formula adopting reduced integration is employed in obtaining the elements of the stiffness matrix. The nonlinear analysis is carried out using a piecewise analysis method by dividing the total load into a finite number of steps and using an iteration process. The following boundary conditions are taken in the analyses:

Simply supported (SS):

$$v=w=\psi_y=0 \quad \text{at } x=0,\ a$$

$$u=w=\psi_x=0 \quad \text{at } y=0,\ b$$

Clamped (C2):

$$w=\psi_x=0 \quad \text{at } x=0,\ a$$

$$w=\psi_y=0 \quad \text{at } y=0,\ b$$

Two opposite edges clamped; the other two simply supported (CSCS):

$$v=w=\psi_y=0 \quad \text{at } x=0,\ a$$

$$w=\psi_y=0 \quad \text{at } y=0,\ b$$

Table 1 shows the values of the central deflection of five- and seven-layered square sandwich plates with SS and C2 boundaries and under uniformly distributed loads of unit intensity. It is observed that the present formulation yields a lower value for the result compared. The variation of linear and nonlinear deflection with the load parameter, $\bar{q}=q_0/E_f(a/h)^4$, where q_0 is the intensity of the uniformly distributed load and E_f is the Young's modulus of the facings, is presented in Fig. 2 for CSCS and C2 boundary conditions. The nonlinear effect is more dominant at higher loads as expected and becomes effective at a lower $\bar{q}$ for a plate with CSCS boundary. The static behavior of a seven-

Table 2 Comparison of natural frequencies

Modal no. m	n	Natural frequencies in present formulation	Hz (7-layered plate)[13]
1	1	19.7	19
2	1	35.5	38
1	2	56.2	60
3	1	71.8	69
2	2	77.7	78
3	3	112.2	109
4	1	115.8	115
1	3	133.2	128

Note: $a = 1828.8$ mm, $b = 1219.2$ mm, $E_f = 7031$ kg/mm^2, $\nu_f = 0.33$, $G_{xzc} = 13.71045$ kg/mm^2, $G_{yzc} = 5.27325$ kg/mm^2, $\zeta_f = 2.8225 \times 10^{-10}$kg s^2/mm^4, $\zeta_c = 1.2423 \times 10^{-11}$ kg s^2/mm^4.

layered sandwich plate under different boundary conditions is shown in Fig. 3. Here again, the plate with CSCS condition indicates higher nonlinearity as against C2 boundary. It is interesting to observe that the nonlinearity effect commences at lower ω/h for the seven-layered case when compared to the five-layered one. In Fig. 4, the comparison of deflections for five- and seven-layered sandwich plates is shown for different boundary conditions. Here, the total thickness of the faces and that of cores in each case are constant. The results indicate that the seven-layered configuration is more flexible.

For the free-vibration case, Eq. (5) takes the form

$$[K]_e\{\Delta e\} = \omega^2 [M]_e \{\Delta\}_e \tag{7}$$

In this case, the frequencies are obtained at a global level by evaluating the eigenvalues. The linear frequencies and the associated eigenvectors are obtained first. The nonlinear frequencies are then obtained by an iteration method. The eigenvectors are multiplied by a specified amplitude ratio for each iteration in order to study the effect of nonlinearity.

The frequencies of a five-layered simply supported rectangular sandwich plate with orthotropic facings for the linear case are compared with those taken from Ref. 13 in Table 2. This shows a good comparison, thereby confirming the acceptability of the element. The ratio of nonlinear to linear frequency and its variation with amplitude ratio is shown in Fig. 5 for symmetrically layered and orthotropic five- and seven-layered square sandwich plates. The total thickness in both cases is assumed to be the same. The thicknesses of facings and cores are calculated individually. It is observed that no change in the frequency ratio occurs until the amplitude ratio is higher than 0.05. However, the deviation occurs at a higher ratio in the case of a five-layered sandwich plate.

Conclusions

The nonlinear bending and vibration behavior of multilayered sandwich plates is studied here, using a quadratic rectangular finite element. Three types of boundary conditions, namely, simply supported, clamped, and a combination of these, are considered for the analysis. Results are presented for five- and seven-layered plates, which indicate that the nonlinearity effect commences at lower deflection for the seven-layered case when compared to the five-layered one. The seven-layered configuration is found to be more flexible than a five-layered sandwich having the same total thickness.

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